UiA University of Agder

Bachelor

Tangent space

Two definitions of tangent spaces on smooth manifolds and their equivalence in differential geometry

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Acknowledgements

I am grateful to my advisor for all the guidance he has given me towards finishing my thesis. I would also like to extend my thanks to Stine and Lennard for providing me with an encouraging working environment. Thank you to my family for their never-ending support, no matter the challenges they themselves are facing. I very much appreciate my therapists' countless hours of listening to me rant about how amazing and stressful this bachelor has been. Lastly, I have to thank my girlfriend for always having my back.

Abstract

This thesis will first defining what a smooth manifold is, then proceed to define tangent space at a point on the manifold as the space of all derivative of all smooth functions and then as an equivalence class of smooth functions going through a point on the manifold. Then prove their equivalence by using an isomorphism.

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Chapter 1

Essential definitions for manifolds

This thesis assumes some prerequisite knowledge before reading, calculus, linear algebra and basic real analysis.

1.1 Topological Manifolds

In this thesis manifolds that are defined in the following manner
Definition
A smooth manifold is a topological space X, that is Hausdorff with a countable basis such that each point $x \in X$ has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$ together with a maximal atlas.

That... is a lot of terms and jargon that needs further clarification to make any sense. So instead of being overwhelmed by more definitions before starting on the main topic of the thesis, equivalence of definition for tangent spaces, we will instead take five steps back and start by looking at the first part of the definition, topology.
The etymology of the word topology is that it is from late Latin topologia, which is from from ancient Greek tópos+(o)logy which means place/locality and study of, respectively [4]. So, topology is "the study of placeness". A common way to introduce topology is that it is the study of objects where you can stretch and bend things to your hearts content, but not tear or puncture your object. This is a somewhat misleading introduction to topology, because topological spaces only refers to open sets, as opposed to stretchiness. However, it is by those open sets that one can define spaces that can be equivelant to each other such that its bended and stretched. But working with open sets is quite different from playdough, so a somewhat more mature introduction is that topological spaces abstracts from metric spaces the notion of closeness. This is also misleading, for there is nowhere in the definition of topological spaces that cares about distance or closeness. It is only when the topological space is defined that one can define that some point is closer than another point. So, the topological space is a set X where the elements are called points and a collection τ_X of subsets that are labeled as open, these open subsets follow the absolute minimum of properties. To clarify, the set $t \in \tau$ is open because it is in τ, and it is in τ because we want it to be open.

Using some well known properties of open sets will give us the framework to define what closeness means. The formal definition of a topological space is as given here:

Definition 1.1.1. A *topological space* is a pair (X, τ) where $\tau \subseteq P(X)$ and

1. $\emptyset, X \in \tau$
2. $\forall \mathcal{U} \subseteq \tau : \bigcup_{U \in \mathcal{U}} U \in \tau$
3. $\forall U_1, U_2 \in \tau : U_1 \cap U_2 \in \tau$

So the topology defines the empty set and the set too to be open. It also gives us that any union of open set is open, even if there are uncountably many! The third axiom tells us that the intersection of two open sets is open, unlike the previous axiom it cannot be uncountable. Rather it has to be a finite collection of sets whose intersection is open.

Example 1.1.1. $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a, b\}, \{b, c\}, \{b\}\}$

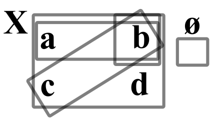

Figure 1.1: The squares are the open sets.

The purpose of a topological space is to take the properties of open sets from metric spaces and define the usual analysis terms, compactness, continuity, completeness, connectedness etc. The metric space originally have open sets to be the set of points who does not contain its border defined using open balls, that again is defined using the metric and inequalities. In that layer of definitions there are certain properties that is redundant for defining the analysis jargon, such as the inequalities. The relevant properties, are the ones given in the definition topological space, that is, the properties of open sets. Therefore, we work with topological spaces, however were going to work with a specific type of topological space, manifolds. An informal description of a manifold, is that it is a topological space that might look reasonable enough in our real life space so that we can do calculus on it. The topological space that will be of most importance in this thesis is the one dubbed the standard topology on $\mathbb{R}^n$.

Definition 1.1.2. The definition consist of two parts, first what a ball is, defined using euclidean metric. Secondly what sets are open, which will be the sets that can be written as an arbitrary union of balls.

1. $B(x, \epsilon) = \{a \in \mathbb{R}^n | \epsilon > \sqrt{\sum_{i=1}^{n} (a_i - x_i)^2}\}$
2. $\tau_{\mathbb{R}^n} = \{U \subset \mathbb{R}^n | U = \bigcup B(x, \epsilon)\}$

Right now we want to show that this specific definition is a topology, however, it is a specific instance of a well known fact that every metric space has an induced topology. Instead of showing the specific instance i will show the more general case. First for the sake of thoroughness, the definition of a metric space.

Definition 1.1.3. A metric space is an ordered pair (M, d) where M is a function and $d : M \to \mathbb{R}^+$ is a function with the following properties.

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \le d(x, y) + d(y, z)$

as promised.

Theorem 1.1.1. *For every metric space (M, d) there is a topological space (M, τ_M) given by the metric.*

$$\tau_M = \{U \subset X | U = \bigcup B_d(x, \epsilon)\}$$

Where $B_d(x, \epsilon) = \{a \in M | d(x, a) < \epsilon\}$

Proof. First of all $\emptyset = \bigcup_{x \in \emptyset} B_d(x, \epsilon)$ and $M = \bigcup_{x \in M} B_d(x, \epsilon)$
Secondly, let us have a subset from τ_M called $\mathcal{U}$, each element is an union of balls, $\forall U \in \mathcal{U} : U = \bigcup B_i(x, \epsilon)$. We now write out that $\mathcal{U} = \bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} (\bigcup B_i(x, \epsilon))$ union of unions is another union $\bigcup B_j(x, \epsilon) \in \tau_M$.
And last, $\forall p \in (\bigcup_{i \in I} B_i(x, \epsilon) \cap \bigcup_{j \in J} B_j(x, \epsilon))$ there exists $B_l(x, \epsilon)$ such that it is an subset of the intersection. So we get that $\bigcup_{l \in L} B_l(x, \epsilon) \in \tau_M$ □

Corollary 1.1.2. $(\mathbb{R}^n, \tau_{\mathbb{R}^n})$ *is a topology*

Now to generate lots of interesting examples of topological spaces we will define the subspace topology.

Definition 1.1.4. For a subset $S \subset X$, the subspace topology (S, τ_S) is defined as

$$\tau_S = \{S \cap U | U \in \tau_X\}$$

Lemma 1.1.3. *Subspace topology is a topology*

Proof. First condition is given by $\emptyset \in \tau_X \implies S \cap \emptyset = \emptyset \in \tau_S$.
Second condition, given some $\mathcal{U} \subset \tau_S$ we have that $\bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} S \cap U_X$ where $U_X \in \tau_X : U_X \cap S = U$. Since $\bigcup_{U \in \mathcal{U}} S \cap U_X = S \cap \bigcup_{U \in \mathcal{U}} U_X$ and $\bigcup_{U \in \mathcal{U}} U_X \in \tau_X$ we have that $\bigcup_{U \in \mathcal{U}} U \in \tau_S$
Third condition, $\forall S_1, S_2 \in \tau_X : (U_1 \cap S) \cap (U_2 \cap S) = (U_1 \cap U_2) \cap S \in \tau_S$ □

Example 1.1.2. The n-sphere $\mathbb{S}^n$

$$\mathbb{S}^n = \{\mathbf{x} | \sum_{i=1}^{n} x_i^2 = r^2\}$$

its topology is the subspace topology $\tau_{\mathbb{S}^n}$ with respect to $\mathbb{R}^{n+1}$

Example 1.1.3. Open triangle.
$T = \{(x, y)|x + y < 1, 0 < x < 1, 0 < y < 1\}$
One can also induce a topology in this subset with the subspace topology.

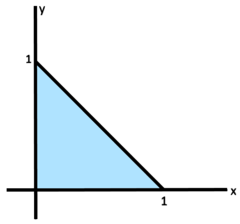

Figure 1.2: Triangle

Example 1.1.4. The parameterization of a torus
$T : [0, 2\pi]^2 \to \mathbb{R}^3$.

$$\text{T(u,v)} = \begin{cases} x = (\cos(u)r + R)\cos(v) \\ y = (\sin(u)r + R)\cos(v) \\ z = r\sin(v) \end{cases}$$

This can also be given the subspace topology.

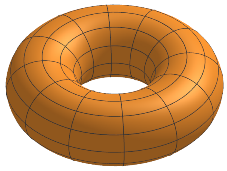

Figure 1.3: Torus with r=1 and R=2

Another way of constructing topological spaces is by taking one space and nicely warp it into another space, and by nicely it is meant that we do not tear or puncture it which is ensured by the second property and the first property ensures that the original space is being bent onto every part of the other space. A concrete example of why the first property is very important is that taking an A4-paper and rolling into a cylinder does not make a torus.

Definition 1.1.5. Let X and Y be topological spaces, $p : X \to Y$ is a *quotient map* if

1. It's surjective
2. If $U \in \tau_Y \iff p^{-1}(U) \in \tau_X$

The second property is something we will get back to later in definition 1.1.13. Since this quotient map is only a certain type of function we need the following definitions to get a topology from a given quotient map.

Definition 1.1.6. Let X be a topology and A a set then the *quotient topology* is defined by a surjective map $p : X \to A$ such that it is a quotient map. That is $\tau_A = \{p^{-1}(U)|U \in \tau_X\}$. This is called the quotient topology induced by p.

Definition 1.1.7. Let X be a topological space and X^* be a set of subsets of X that are disjoint and whose union is X. The *quotient space* (X^*, τ_{X^*}) is a quotient topology induced by p, such that p maps each point in X to its corresponding set in X^*.

The quotient topology can be easily used to construct a topology for the famous Möbius strip.

Example 1.1.5. Taking the unit square $I^2 = [0, 1]^2$ and the mapping $\rho : I^2 \to \mathcal{M}$

$$\rho(x, y) = \begin{cases} (0, 1 - y) & \text{If } x = 1 \\ (x, y) & \text{otherwise} \end{cases}$$

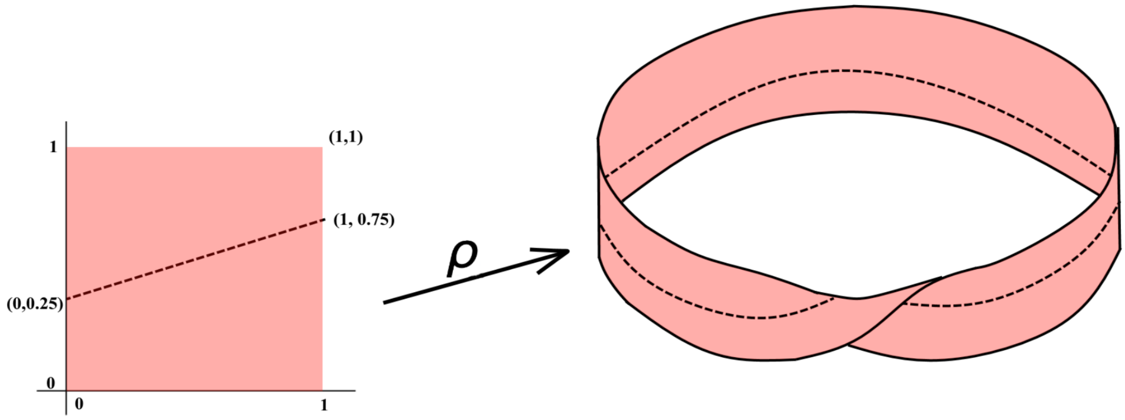

Figure 1.4: Möbius strip with dotted path

we get that $X^* = \{\rho^{-1}(p) | \forall p \in \mathcal{M}\}$
more explicitly $\{\{(x,y)\} | 0 < x < 1 \text{ and } 0 \leq y \leq 1\} \cup \{(0, 1-y), (1, y)\} | x = 1 \text{ and } 0 \leq y \leq 1\}$.
$\tau_{X^*} = \{\rho^{-1}(U) | U \in \tau_{I^2}\}$. Where I^2 has the subspace topology as a subspace of $\mathbb{R}^2$.

One can also construct a topology for the Klein bottle using the quotient topology.

Example 1.1.6. Similarly to the Möbius strip the quotient map is $\kappa : I^2 \to \mathcal{K}$

$$\kappa(x,y) = \begin{cases} (0, 1-y) & \text{If } x = 1 \\ (x, 0) & \text{If } y = 1 \\ (x, y) & \text{otherwise} \end{cases}$$

which creates the quotient space $(\mathcal{K}, \tau_{\mathcal{K}})$

Before continuing to relevant parts of the definition of manifolds we take some important deviations from defining manifolds to define what it means for two points in a topology to be closer to each other than some other pair of points and what is meant for with a neighborhood.

Definition 1.1.8. A set C is *connected* if there does not exist two open sets A and B such that they are disjoint and cover C

Definition 1.1.9. A *open neighborhood* of a point $p \in X$ is a set U such that $p \in U$ and $U \in \tau$

Definition 1.1.10. Given three points a,b,c in a open and connected set U, we have that if there is a open and connected set V containing a and b but not c then b is *closer* to a than c.

This notion of closer is not something that is guarenteed to exists, because there are topological spaces where there are no connected sets except sets of a single point. However, constructing convergent sequences is always possible. One of the properties we want is unique convergence, because anything else is unfamiliar, unforgivably confusing and for most people just a pedantic counter examples. There is a quote from Penrose "I must ... return firmly to sanity by repeating three times: spacetime is a Hausdorff differentiable manifold; spacetime is a Hausdorff ...". Penrose attempted to develop spacetime as a non-Hausdorff manifold. [1] One easy example of non-Hausdorff topological space is the line with two origins.

Example 1.1.7. *Line with two origins.* $\mathcal{N} = \mathbb{R} \setminus \{0\} \cup \{p\} \cup \{q\}$. Where the topology $\tau_{\mathcal{N}}$ is the subspace topology with some additional sets. These sets are on the form $(a, 0) \cup \{p\} \cup (0, b)$ and $(a, 0) \cup \{q\} \cup (0, b)$ where $a < 0$ and $0 < b$.

Now we shall define convergence and follow up with a non-example of unique convergence on the line with two origins.

Definition 1.1.11. Given a sequence $\{p_n\}_{n=1}^{\infty}$ of points in X and a point $p \in X$, the sequence is said to converge to p if for every neighboorhoood U of P, there exists a positive integer N such that $p_n \in U$ for all $n \geq N$. This is denoted $\lim\limits_{n\to\infty} p_n = p$

Example 1.1.8. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ of points in the line with two origins $\mathcal{N}$ converges both to p and q.

The first property of topological spaces that are smooth is that it is a Hausdorff space.

Definition 1.1.12. A topological space (X, τ) is a *Hausdorff space* when

$$\forall p, q \in X : p \neq q \exists \text{ disjoint neighborhoods } U \text{ and } V \text{ for each point } p \text{ and } q, \text{ respectively.}$$

In other words, for any two points there is open set containing them that are disjoint. This property gives us unique convergence.

Theorem 1.1.4. *Hausdorff space has unique convergence.*

Proof. Given a sequence $S = \{s_n\}_{n=1}^{\infty}$, suppose there were two distinct limits l and m such that $l \neq m$. We then have from the Hausdorff property that there exists two open neighborhoods U_l and U_m such that $U_l \cap U_m = \emptyset$. We also have from the definition of convergence that $\exists N_{U_l} \in \mathbb{R} : n > N_{U_l} \implies x_n \in U_l$ and $\exists N_{U_m} \in \mathbb{R} : n > N_{U_m} \implies x_n \in U_m$. Letting $N = \max(N_{U_l}, N_{U_m})$ we then get $\exists N \in \mathbb{R} : n > N \implies x_n \in U_m$ which contradicts that $U_l \cap U_m = \emptyset$. □

The next step is to define homeomorphic, which we will use to make the space reasonable looking. Were going to use homeomorphism to make sets in our manifold homeomorphic to sets of the real numbers or more formally to open sets in the topology for $\mathbb{R}^n$, with the standard topology. This can be generalized by considering topological vector spaces instead, which is outside the scope of this thesis. Homeomorphisms are to be thought naively of as deformations shapes, where one does not tear, cut or poke holes. The most well known example of homeomorphism is that coffee cups and donuts are homeomorphic.

Definition 1.1.13. A function $f : X \to Y$ is *continuous* if $\forall U \in \tau_Y, f^{-1}(U) \in \tau_X$

Example 1.1.9.

We can construct a continuous function with $f(x) = \begin{cases} x & \text{If } 0 \leq x \leq 1 \\ x-1 & \text{If } 2 < x \leq 3 \end{cases}$ is continuous with respect to the subspaces topologies $\tau_{[0,1]\cup(2,3]}$ and $\tau_{[0,2]}$. However, the inverse $f^{-1}(x) = \begin{cases} x & \text{If } 0 \leq x \leq 1 \\ x+1 & \text{If } 1 < y \leq 2 \end{cases}$ is not continuous because $(0.5, 1] \in \tau_{[0,1]\cup(2,3]}$ and $f((0.5,1]) = (0.5,1] \notin \tau_{[0,2]}$

Example 1.1.10. Let $\mathcal{D}$ be any nonempty set and the topology $\tau_{\mathcal{D}}$ be all subsets of $\mathcal{D}$. This is called the *discrete topology.* Any function $f : \mathcal{D} \to X$ where X is a topological space, is continuous.

Definition 1.1.14. That two sets are *homeomorphic* means that there exists an homeomorphism. A homeomorphism is a functions $f : X \to Y$ where

1. f is bijective
2. f is continuous
3. f^{-1} is continuous

Everything is continuous with the discrete topology. So to make sure topologies are not to big, one has a countable basis.

Definition 1.1.15. A *countable basis* $\mathcal{B}$ for a topological space (X, τ) is a subset of the topology such that:

1. $\forall x \in X : \exists B \in \mathcal{B} : x \in B$
2. For any pair $A, B \in \mathcal{B}$ we have that $\forall x \in A \cap B$ there exists another basis element C such that $x \in C \subseteq A \cap B$
3. $\mathcal{B}$ is countable

The most common countable basis is the basis for the real numbers.

Example 1.1.11. $\mathcal{B}_{\mathbb{R}} = \{(r,s) | r, s \in \mathbb{Q}\}$ is a basis for the standard topology.

Now we have all the properties of a topological manifold defined, putting it all togheter we can finally write out the definition knowing what it means.

Definition 1.1.16. A *topological m-mainfold* is a Hausdorff space X with a countable basis $\mathcal{B}$ such that each point $x \in X$ has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$ with the standard topology.

This is a definition that is studied in its own right, as far as I know, the subject does not have a name other than "topological manifolds". However, smooth manifolds is studied in the subject differential geometry. The difference between topological and smooth manifolds are that there are further restrictions on what can be a smooth manifold. These restrictions are referred to as a smooth structure, because it demands the existence of some functions with the manifold as its domain.

1.2 Smooth structure on a topological manifold

The way this smooth structure is constructed is as follows: first what a chart is, then smoothly compatible charts, then a collection of smoothly compatible charts called an atlas. The reason why we want charts is because addition is not necessarily defined on topological spaces, "Oslo" + "London" does not make sense. However, we have made maps of our planet, which is used to talk about mathematical operations. For examples latitudes and longitudes is a chart of the earth, and Oslo has coordinates $(59.911491, 10.757933)$ and London has $(51.509865, -0.118092)$ which when we subtract them gives us $(8.401626, 10.876025)$. The difference between London and Oslo tells us the distance in terms of longitude and latitude. This demonstrates unnecessarily concretely that charts are essential to do calculus.

Definition 1.2.1. A d-dimensional *chart* of M is an ordered pair (U, φ), where U is an open subset of M $\varphi : U \to D$ is a homeomorphism of U onto an open subset D of $\mathbb{R}^n$ with the standard topology.

The inverse of parameterization of manifolds are an example of charts, its the inverse since the domain is the respective manifold. But it is not any chart that we are interested in, such as charts from a triangle to the line. Since the triangle is pointy at its corners we cannot do calculus at that point since there are no single tangent line for derivatives. However, there does exists charts from the corner of a triangle that simply straightens it out into a line. The clue lies in the transition between maps. Before google maps people navigated completely manually using maps, aka charts. These maps have overlap which may seem like a waste of space but was rather important when moving from one map to another to not loose ones position. This overlap of maps is called a transition map.

Definition 1.2.2. Let (U, φ) and (V, ψ) be d-dimensional charts of M. Let $U \cap V \neq \emptyset$. The *transition map* from φ to ψ is the mapping: $\varphi \circ \psi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$

Notice that this is a function between spaces where one can already do calculus. This is where most types of manifolds differ, in the type of properties this transition map has. Recalling the definition of smooth from analysis.

Definition 1.2.3. $f : \mathbb{R}^d \to \mathbb{R}^m$ is *smooth* if $\forall n \in \mathbb{N}$ we have that $\dfrac{\partial^n f}{\partial x_i^n}, \forall 1 \leq i \leq d$ and is continuous. The set of all smooth functions is $C^\infty(\mathbb{R}^d, \mathbb{R}^m)$

Example 1.2.1.
There are many familiar smooth functions, to list a few:

1. Trigonometric functions $\sin, \cos$ and $\tan$
2. Polynomials of one or more variables
3. $\exp(x) = e^x$ and $\ln(x)$

Composition, multiplication and addition of smooth functions are smooth from the properties of derivatives.

Definition 1.2.4. (U, φ) and (V, ψ) are *smoothly compatible* if and only if their transition mapping is of class C^∞.

If one wants a different manifold kind of manifold one changes the compatibility condition. For example if one wants a C^k-manifold one demands that they are C^k compatible. If one wants a complex manifold one has a holomorphic compatible charts.

Definition 1.2.5. Given a topological space M a C^∞ *atlas* is a collection of charts $\{\varphi_i : U_i \to \mathbb{R}^n\}_{i \in I}$ such that any two charts are smoothly compatible.

There are uncountable infinite many different smooth atlases. However we want to pick the best one, and that is the one where all the smoothly compatible charts are.

Definition 1.2.6. A smooth atlas $\mathcal{A}$ is a maximal smooth atlas on a manifold M is maximal if it is not properly contained in any larger smooth atlas.

Lemma 1.2.1. *Given a maximal smooth atlas $\mathcal{A}$ on a manifold M there does not exists a any smoothly compatible chart that is not contained it.*

Proof. If there was a smoothly compatible chart not in $\mathcal{A}$ then it would be contained in another smooth atlas with that chart so it is not maximal. Which contradicts the premise. □

Definition 1.2.7. A smooth manifold is a quadruple $(M, \tau_M, \mathcal{B}, \mathcal{A})$ where M is a set, τ_M is a Hausdorff topology, $\mathcal{B}$ is a countable basis for the given topology and $\mathcal{A}_M$ is a maximal smooth atlas.

The rest of the text is not going to specify the topology, basis or the atlas. Rather simply referring to "a manifold M" and imply the other structure.

Example 1.2.2. The euclidean space is a smooth manifold. Where the atlas is charts $(U, id_U(\vec{x}))$

1.3 Smooth maps

Extending definition of smooth to maps between manifolds.

Definition 1.3.1. A function $f : M \to N$ *between the manifolds* M and N is *smooth* if and only if for every pair of charts $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ we have that $\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth. The set of all smooth functions between them is denoted $C^\infty(M, N)$

Homeomorphisms are used to classify topological spaces. Now with smooth functions between smooth manifolds we can make an analogous definition.

Definition 1.3.2. Two smooth manifolds M and N are *diffeomorphic* if there exists a *diffeomorphism*, which is a function $F : M \to N$ that is bijective, and where both F and F^{-1} are smooth.

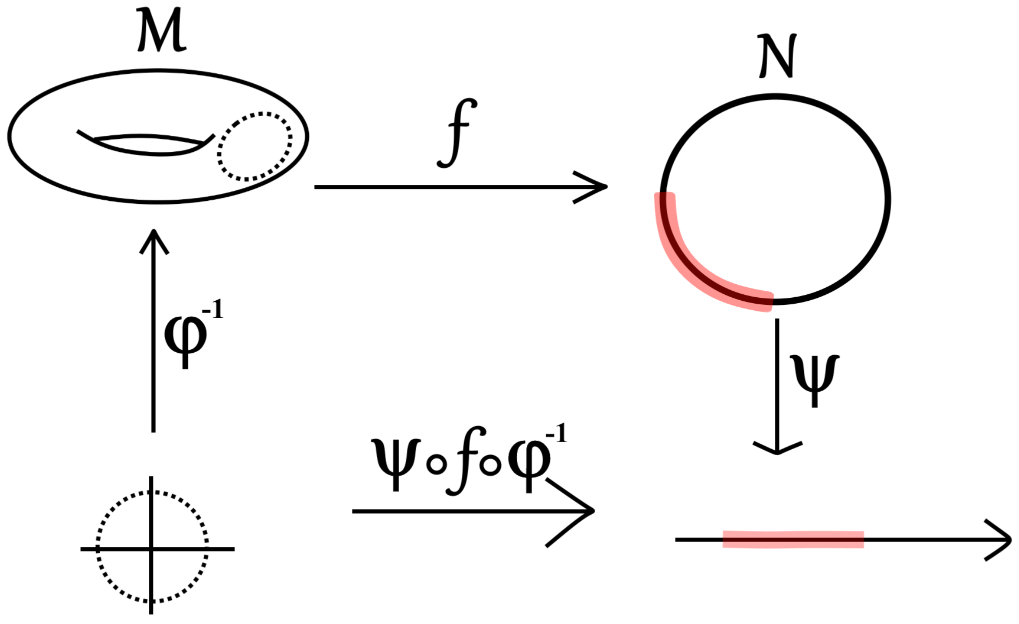

Figure 1.5: Smooth map

Example 1.3.1. $F : \mathbb{S}^1 \to$ ellipse, $F(x, y) = (ax, by)$ and $F^{-1} = (\frac{x}{a}, \frac{y}{b})$

There is a trivially common diffeomorphism, charts.

Lemma 1.3.1. *Charts are diffemorphisms*

Proof. Given a chart $\xi : M \to \mathbb{R}^n$ for it to be diffeomorhic $\psi \circ \xi \circ \phi^{-1} : \phi(U \cap \xi^{-1}(V)) \to \psi(V)$ for all pair of charts. Inserting that the charts for the standard euclidean space is the identity maps we get $id \circ \xi \circ \phi^{-1} : \phi(U \cap \xi^{-1}(V)) \to id(V) = \xi \circ \phi^{-1} : \phi(U \cap \xi^{-1}(V)) \to V$ □

Whenever $\mathbb{R}^n$ is used in combination with a manifold M, for example $C^\infty(M, \mathbb{R})$, it is implied that the standard topology and the atlas in 1.2.2 is used. That a function f is in $C^\infty(M, \mathbb{R})$ means that for every chart $(U, \phi) \in \mathcal{A}_M$ we have that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth.

Chapter 2

Tangent spaces

Now that we have defined manifolds and seen some of them, the thing we need know is tangent spaces. This is the essential thing we want from euclidean space. As a curves' derivative is given by a tangent line and a surfaces' derivative is obtained with a tangent plane, both which is vector spaces. As we have generalized linear structures to vector spaces and we want to generalise calculus to use more of the abstract tools from linear algebra.

2.1 Geometric tangent space

Geometric tangent space is a simple approach to construct a tangent

Definition 2.1.1. *Geometric tangent space* is defined by $\mathbb{R}^n_a = \{a\} \times \mathbb{R}^n = \{(a, v) : \vec{v} \in \mathbb{R}^n\}$ where $\vec{v}_a$ denotes a vector in $\mathbb{R}^n_a$

$\mathbb{R}^n_a$ is a vector space under the addition operation $\vec{v}_a + \vec{w}_a = (\vec{v} + \vec{w})_a$. It has to be remarked this vector space exists separate from the space it is tangent too. In fact all tangent spaces are disjoint, and the illustration of a line tangent to a curve is a simplified drawing. btw $\mathbb{R}^n_a$ isomorphic to $\mathbb{R}^n$

This does not give us any insight about the geometry of the manifold. Just slapping a vector space on each point is quite useless, because the lack of precision of mentally throwing vector spaces on topological spaces has no precision. There is no way determine if it is tangent. Take a curve $y = x^2$ for example, just saying there is a vector space $\mathbb{R}_{(2,4)}$ is not insightful. The way we determine this vector space is that is the span of a vector which is the instantaneous rate of change at that point. Some quick calculus says that this tangent space is $\{(2, 4)\} \times span([1, 4])$

Do not loose hope. There is a result about directional derivatives that gives us a hint to relate vectors and derivatives. So if we can extend derivatives to manifolds we might be able to use the relation between directional derivatives and vectors.

Recalling the definition of the directional derivatives.

Definition 2.1.2. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector $\vec{v}$ then the directional derivative

$\nabla_{\vec{v}} : \{f | f : \mathbb{R}^n \to \mathbb{R}\} \to \mathbb{R}$ is

$$\nabla_{\vec{v}} f = \lim_{h \to 0} \frac{f(\vec{x} + h \cdot \vec{v}) - f(\vec{x})}{h}$$

The space of all directional derivatives is $\Delta^n = \{\nabla_{\vec{v}} | \vec{v} \in \mathbb{R}^n\}$

Theorem 2.1.1. *If f is differentiable then $\nabla_{\vec{v}} f = \nabla f \bullet \vec{v}$*

Proof. We begin by writing out that $\nabla f(\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\vec{x}) \vec{e}_i$
Looking at the definition $\lim_{h \to 0} \frac{f(\vec{x}+h \cdot \vec{v}) - f(\vec{x})}{h}$ we see that if we define a function $\gamma(t) = \vec{x} + t\vec{v}$ that has the properties $\gamma(0) = \vec{x}$ and $\frac{\partial}{\partial t}\gamma(t) = \vec{v}$. Then we can rewrite it as $\lim_{h \to 0} \dfrac{f(\gamma(h)) - f(\gamma(0))}{h} = \frac{\partial}{\partial t} f(\gamma(t))|_0$ we then use the multi-variable chain rule $\frac{\partial}{\partial t} f(\gamma(t))|_0 = \sum_{i=1}^{n} \dfrac{\partial f}{\partial \gamma_i}\Big|_0 \dfrac{\partial \gamma_i}{\partial t}\Big|_0 = \sum_{i=1}^{n} \dfrac{\partial f}{\partial x^i} v^i = \nabla f \bullet \vec{v}$ □

Corollary 2.1.2. *$F : \mathbb{R}^n_p \to \Delta^n$ defined by $F(\vec{e}_i) = \frac{\partial}{\partial x^i}|_p$ is an isomorphism*

Proof. Both vector spaces are finite dimensional, of same dimension and F is bijective between the basis vectors then F is a isomorphism. □

2.2 Derivation

Generalising derivatives to smooth functions on manifolds

Definition 2.2.1. Given a manifold $M \supset U$, a derivation ω on smooth functions is a function $\omega : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R})$ such that

1. $\omega(af + bg) = a\omega(f) + b\omega(g)$
2. $\omega(fg) = \omega(f) \cdot g + f \cdot \omega(g)$

$Der(U) = \{\omega | \omega \text{ is a derivation on } U\}$

This definition is chart independent!

Lemma 2.2.1. *$Der(U)$ is a vector space over $\mathbb{R}$*

Proof. Let $\omega, \alpha \in Der(U)$
$(\omega + \alpha) \in Der(U)$ because linear combination of linear functions is linear.
$(\omega + \alpha)(fg) = \omega(fg) + \alpha(fg) = \omega(f)g + f\omega(g) + \alpha(f)g + f\alpha(g) = g(\omega(f) + \alpha(f)) + f(\alpha(g) + \omega(g)) = g((\omega + \alpha)(f)) + f((\alpha + \omega)(g))$, this implies that $(\omega + \alpha)$ is Leibniz. Since sum of two derivations are a linear and Leibniz its a derivation and it is therefore closed under addition by the remark that the definition says all derivations.
$\omega + \mathbf{0} = \omega$ exists defined by $Der(U) \ni \mathbf{0}(f) = 0$. The rest of the criteria for vector spaces is proved by rearranging a lot of parenthesises. □

Definition 2.2.2. Let $U \subset M$ of a n-dimensional manifold. The *evaluation function* $|_p : C^\infty(U, \mathbb{R}) \to \mathbb{R}$ is a function that evaluates $f : U \to \mathbb{R}$ at $p \in M$, defined by $f|_p = f(p)$ and $\omega f|_p = f'(p)$ where $f' = \omega(f)$.

$\omega((f)|_p) = 0$ vs $(\omega(f))|_p = f'(p)$

Note that for the Lebniz rule it distributes like this $\omega(fg)|_p = (\omega(f)g + f\omega(g))|_p = \omega(f)|_p \cdot g(p) + f(p) \cdot \omega(g)|_p$. If we let the evualution function act on derivations, some magic happens. $(a\omega + b\alpha)|_p = a(\omega)|_p + b(\alpha)|_p$ Because it is linear and a function between two vector spaces. $|_p : Der(U) \to \mathbb{R}$ then the set of all this linear maps form a vector space.

Definition 2.2.3. The tangent space T_pM to a manifold M is defined by

$$T_pM = \{\omega|_p | \omega \in Der(U), p \in U\}$$

Tangent vectors in T_pM will be denoted with X, Y and D

That is, we define tangent vectors to be derivations **evaluated** at p. We are now going to find the basis for this vector space. First we are going to prove an isomorphism between n-dimensional directional derivatives Δ^n_f and tangent space T_pM where $M = \mathbb{R}^n$. That is we will show that $\Delta^n_f \cong T_p\mathbb{R}^n$. Then we will show for any n-dimensional manifold M that $T_pM \cong T_p\mathbb{R}^n$

Lemma 2.2.2.
For all $D \in T_pM$ we have that
(i) $f = constant \implies D(f) = 0$
(ii) $f(p) = g(p) = 0 \implies D(fg) = 0$

Proof. (i) $D(c) = D(c \cdot 1) = D(c) \cdot 1 + c \cdot D(1) = 2D(c)$ this implies that $D(c) = 0$.
(ii) $D(fg) = D(f)g + fD(g) = 0g + f0 = 0$ □

Lemma 2.2.3. *For every $D \in T_pM$ there exists a scaled tangent vector $X \in T_pM$ such that $D(x^i) = aX(x^i)$ and $X(x^i) = 1$.*

Proof. $X(x^i) = \frac{D(x^i)}{a}$ and since it is a vector space it is closed under scalar multiplication. □

For every geometric tangent vector at p we have a directional derivative at p
For the next theorem we will need Hademars lemma. Most text use Taylors theorem with reminder.

Lemma 2.2.4. *Hademars lemma: Let f be a smooth, real-valued function defined on an open neighborhood U of a point $\vec{a}$ in $\mathbb{R}^n$. Then for all $\vec{x} \in U$*

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} (x^i - a^i) g^i(\vec{x})$$

Where

$$g_i(\vec{x}) = \int_0^1 \frac{\partial f}{\partial x^i}(\vec{a} + t(\vec{x} - \vec{a}))dt, \text{ and } g^i(\vec{a}) = \frac{\partial f}{\partial x^i}|_{\vec{a}}$$

We can now prove that derivations and directional derivatives are not only isomorphism but actually equal for $\mathbb{R}^n$. similar to what we did earlier but we do not know dimension so we cannot rely on that.

Theorem 2.2.5. *$\Delta_p^n = T_p\mathbb{R}^n$, where $\Delta_p^n = \{\nabla_{\vec{v}}|_p | \vec{v} \in \mathbb{R}^n\}$*

Proof. Every directional derivative is a derivations so $\Delta_p^n \subset T_p\mathbb{R}^n$. Showing that every derivation $D \in T_p\mathbb{R}^n$ evaluated at p is a linear combination of partial derivatives evaluated at p as shown by the following computation is sufficient to show that every derivation is a directional derivative.

$D(f)$	Hademars lemma
$= D\left(f(\vec{p}) + \sum_{i=1}^n (x^i - p^i)g^i(\vec{x})\right)$	Linear
$= D(f(\vec{p})) + \sum_{i=1}^n D\Big((x^i - p^i) \cdot g^i(\vec{x})\Big)$	$D(Constant) = 0$ and Leibniz
$= 0 + \sum_{i=1}^n D(x^i - p^i) \cdot g^i(\vec{x})\|_{\vec{p}} + (x^i - p^i)\|_{\vec{p}} \cdot D(g^i(\vec{x}))$	Evaluating the evaluation function
$= \sum_{i=1}^n D(x^i - p^i)g^i(\vec{p}) + 0 \cdot D(g^i(x))$	Linear and times zero is zero
$= \sum_{i=1}^n (D(x^i) - D(p^i))g^i(p)$	$g^i(\vec{p}) = \frac{\partial f}{\partial x^i}$ and derivation of constant is zero
$= \sum_{i=1}^n D(x^i)\frac{\partial f}{\partial x^i}\|_{\vec{p}}$	lemma 2.2.3 where $D(x^i) = v^i X(x^i)$
$= \sum_{i=1}^n v^i X(x^i)\frac{\partial f}{\partial x^i}\|_{\vec{p}}$	$X(x^i) = 1$ and $\nabla_{\vec{v}} f\|_{\vec{p}} = \sum v^i \frac{\partial}{\partial x^i} f\|_{\vec{p}}$
$= \nabla_{\vec{v}} f\|_p$	

Therefore is every derivation a directional derivative $\forall D \in T_p\mathbb{R}^n \; D = \nabla_{\vec{v}}|_p$. So we have that $\Delta_f^n \supset T_p\mathbb{R}^n$. □

So far we have that $T_p\mathbb{R}^n = \Delta^n \cong \mathbb{R}_a^n$. To prove that $T_pM \cong T_p\mathbb{R}^n$ for a n-dimensional manifold we need a new tool called the differential.

Definition 2.2.4. A *Differential* is a function $d : C^\infty(M, N) \to hom(T_pM, T_qN)$ Where $F \in C^\infty(M, N), f \in C^\infty(N, \mathbb{R}^n)$, $q = F(p)$ and $D \in T_pM$, We have that $dF_p : T_pM \to T_qN$

is defined by $dF_pD(f) = D(f \circ F) = D(f(F))$

Lemma 2.2.6. *Let M be a smooth manifold, $p \in M$ and $D \in T_pM$.*
Also given that $f, g \in C^\infty(M, \mathbb{R})$, where $g(x) = f(x), \forall x \in U$ for some neighborhood U of p. We have that $D(f) = D(g)$

Proof. First define $h = f - g$, h is smooth. $h(p) = f(p) - g(p) = 0$. Let ψ be a smooth bump function (for their existence see [2] page 44) where $\psi(p) = 0$ and $\psi(x) = 1, \forall x : h(x) > 0$. $\psi(p) = 0$ and $h(p) = 0 \xRightarrow{\text{Lemma 2.2.2}} D(h \cdot \psi) = D(h) = 0$. Since $D(h) = 0$ we get that $D(h) = D(f - g) = D(f) - D(g) = 0 \implies D(f) = D(g)$. (page 56 [2]) □

The following lemma is needed because $D \in T_pM$ is defined for some $U \in \tau_M$

Lemma 2.2.7. *Let M be a smooth manifold, let $U \subset \tau_M$, and let $\iota : U \to M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.*

Proof. proof on page 56 [2] □

The next result says that diffeomorphic manifolds have isomorphic tangent spaces, which makes sense because manifolds are locally euclidean and they have to be of the same dimension to be diffeomorphic, and small patches of euclidean spaces surely must have the same tangent space. Though the proof follows a simpler and rigorous line of though.

Theorem 2.2.8. *Given $F : M \to N$ diffeomorphism then $dF : T_pM \to T_pN$ is an isomorphism*

Proof. Given $F \in C^\infty$ and $F^{-1} \in C^\infty$ and $F(F^{-1}) = F^{-1}(F) = id$.

$$\begin{aligned} dF_p(aX + bY) &= (aX + bY)(F(f)) \\ &= aX(F(f)) + bY(F(f)) \\ &= adF_p(X) + bdF_p(Y) \end{aligned}$$

Let $X \in T_pM$ and $Y \in T_qN$, now have to show that $dF_p^{-1}(dF_p(D)) = id = dF_p(dF_p^{-1}(D))$

$$\begin{aligned} &(dF_p \circ dF_p^{-1})(D) \\ &= (dF_p(dF_p^{-1}(D))) \\ &= dF_p(D(F^{-1}(f)))\Big| g = D(F^{-1}(f)) \\ &= dF_p(g) = g(F(f)) \\ &= D(F^{-1}(F(f))) \\ &= D(f) \\ &= id(D) \end{aligned}$$

Same line of argument for $dF_p(dF_p^{-1}(D)) = id(D)$ □

Corollary 2.2.9. *For a chart (U, ϕ), $d\phi : T_pM \to T_p\mathbb{R}^n$ is an isomorphism.*

Proof. We have that $\phi : M \to \mathbb{R}^n$ is a diffemorphism from 1.3.1. The preceding theorem 2.2.8 says that differential of diffeomorphisms are isomorphisms. $\square$

We now have that $T_pM \cong T_p\mathbb{R}^n = \Delta_p^n \cong \mathbb{R}_p^n$. Now lets calculate what the basis for a tangent space for an arbitrary smooth manifold.

Theorem 2.2.10. $T_pM = span(\{\frac{\partial\phi^{-1}}{\partial x^i}|_p\}_{i=1}^n)$ *where n is the dimension of M*

Proof. Let $D \in T_pM$, $\nabla_{\vec{v}}|_p \in T_p\mathbb{R}^n$ and $\phi^{-1} : \mathbb{R}^n \to M$. We have that $D = d\phi_p^{-1}\nabla_{\vec{v}}|_p = \nabla_{\vec{v}}|_p(f \circ \phi^{-1}) = \sum_{i=1}^n v^i \frac{\partial f(\phi^{-1})}{\partial x^i}|_{\hat{p}}$

$\square$

Example 2.2.1. Let $M = \mathbb{S}^1$ and $p = (0, 1)$
$\phi^{-1}(\theta) = (\cos(\theta), \sin(\theta))$ $D \in T_p\mathbb{S}^1$ we now know that $D(f) = a\frac{\partial f(\phi^{-1})}{\partial\theta}|_p$. If $f(x, y) = x + y$ then $f(\phi^{-1}) = \cos(\theta) + \sin(\theta)$ and $f' = \cos(\theta) - \sin(\theta)$
$f'|_{\frac{\pi}{2}} = -1$ and $D(f) = -a \cong ae_1$

2.3 Equivalence class

going to define an equivelance class for which will be used to give another definition of tangent space. too distinguish from earlier tangent space they will have a tilde over them.

Definition 2.3.1. An equivalence relation $\sim$ is a relation that is
Reflexive $a \sim a\ \forall a$
Symmetric $a \sim b \implies b \sim a$
Transitive $a \sim b$ and $b \sim c \implies a \sim c$

Definition 2.3.2. An equivelance class is a set $[a] = \{b | b \sim a\}$. That is, it is the set of all elements equivalent to the required.

Lemma 2.3.1. *Equivalence class is independent of representative of the class, that is $a, b \in [a] \implies [a] = [b]$*

Proof. This is true because $\sim$ is symmetric $\square$

Definition 2.3.3. Let $I = (a, b)$ where $a < 0 < b$. A *tangent vector* is the equivalence class of smooth curves $\eta, \gamma : I \to M$ where $\gamma(0) = \eta(0) = p$ with equivalence relation

$$\frac{\partial}{\partial t}(\phi \circ \gamma_1)|_0 = \frac{\partial}{\partial t}(\phi \circ \gamma_2)|_0$$

for all $\phi \in \mathcal{A}$

Example 2.3.1. Given $M = \mathbb{S}^2, p = (\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}), \hat{p} = (\frac{\pi}{3}, \frac{\pi}{4})$ and $\gamma_1(t) = (\cos(t), \sin(t))$

Definition 2.3.4. Tangent space second edition is defined by $\widetilde{T_p}M = \{[\gamma] | \gamma(0) = p\}$

We now will show that $\widetilde{T}_pM$ is a vector space. Where the addition and scaling is defined with the help of charts, since adding curves point wise gives points outside the manifold. In addition when adding $\phi(\gamma) + \phi(\eta)$ we first of all have a function that goes from $\mathbb{R} \to \mathbb{R}^n$ instead of $\mathbb{R} \to M$, and at zero it gives $2\phi(p)$, this is fixed by using the inverse chart and subtracting $\phi(p)$. In a similar manner when scaling we first use a chart $a\phi(\gamma)$, but then we have $a - 1$ too many $\phi(p)$. This is easily fixed as by subtracting $(a - 1)\phi(p)$.

Theorem 2.3.2. *Tangent space second edition is a vector space.*
Where addition is defined as

$$[\gamma] + [\eta] = [\phi^{-1}(\phi(\gamma) + \phi(\eta) - \phi(p))]$$

and scalar multiplication is defined as

$$a \cdot [\gamma] = [\phi^{-1}(a\phi(\gamma) - (a - 1)\phi(p))]$$

Proof. First the addition axioms. Closed under addition, $[\gamma] + [\eta] = [c] \in \tilde{T}_pM$. This comes from the fact that $c(0) = \phi^{-1}(\phi(\gamma(0)) + \phi(\eta(0)) - \phi(p)) = \phi^{-1}(\phi(p) + \phi(p) - \phi(p)) = p$ and that the composite of smooth functions are smooth and therefor the sum is smooth.
It is associative, $([\gamma] + [\eta]) + [c] = [\gamma] + ([\eta] + [c])$. Because,

$$\begin{aligned}
&([\gamma] + [\eta]) + [c] \\
&= [\phi^{-1}(\phi(\gamma) + \phi(\eta) - \phi(p))] + [c] \\
&= [\phi^{-1}(\phi(\phi^{-1}(\phi(\gamma) + \phi(\eta) - \phi(p))) + \phi(c) - \phi(p))] \\
&= [\phi^{-1}(\phi(\gamma) + \phi(\eta) - \phi(p) + \phi(c) - \phi(p))] \\
&= [\phi^{-1}(\phi(\gamma) + \phi(\eta) + \phi(c) - \phi(p) - \phi(p))] \\
&= [\phi^{-1}(\phi(\gamma) + \phi(\phi^{-1}(\phi(\eta) + \phi(c) - \phi(p))) - \phi(p))] \\
&= [\gamma] + [\phi^{-1}(\phi(\eta) + \phi(c) - \phi(p))] \\
&= [\gamma] + ([\eta] + [c])
\end{aligned}$$

It is commutative

$$\begin{aligned}
&[\gamma] + [\eta] \\
&= [\phi^{-1}(\phi(\gamma) + \phi(\eta) - \phi(p))] \\
&= [\phi^{-1}(\phi(\eta) + \phi(\gamma) - \phi(p))] \\
&= [\eta] + [\gamma]
\end{aligned}$$

Sum identity $= [p]$

$$\begin{aligned}
&[\gamma] + [p] \\
&= [\phi^{-1}(\phi(\gamma) + \phi(p) - \phi(p))]
\end{aligned}$$

$$= [\phi^{-1}(\phi(\gamma)]$$
$$= [\gamma]$$

Sum inverse, $[\gamma] + [\eta] = [p], \eta = \phi^{-1}(2\phi(p) - \phi(\gamma))$
$\eta(0) = p$ so it is well defined.

$$\begin{aligned}
&[\gamma] + [\phi^{-1}(2\phi(p) - \phi(\gamma)]\\
&= [\phi^{-1}(\phi(\gamma) + \phi(\phi^{-1}(2\phi(p) - \phi(\gamma)) - \phi(p))]\\
&= [\phi^{-1}(\phi(\gamma) + 2\phi(p) - \phi(\gamma) - \phi(p))]\\
&= [\phi^{-1}(\phi(p))]\\
&= [p]
\end{aligned}$$

Compatibility $a(b \cdot [\gamma]) = (ab) \cdot [\gamma]$

$$\begin{aligned}
&a(b \cdot [\gamma])\\
&= a[\phi^{-1}(b\phi(\gamma) - (b-1)\phi(p))]\\
&= [\phi^{-1}(a\phi(\phi^{-1}(b\phi(\gamma) - (b-1)\phi(p))) - (a-1)\phi(p))]\\
&= [\phi^{-1}(a(b\phi(\gamma) - (b-1)\phi(p)) - (a-1)\phi(p))]\\
&= [\phi^{-1}(ab\phi(\gamma) - a(b-1)\phi(p) - (a-1)\phi(p))]\\
&= [\phi^{-1}(ab\phi(\gamma) - (a(b-1) + (a-1))\phi(p))]\\
&= [\phi^{-1}(ab\phi(\gamma) - (ab - a + a - 1)\phi(p))]\\
&= [\phi^{-1}(ab\phi(\gamma) - (ab-1)\phi(p))]\\
&= (ab) \cdot [\gamma]
\end{aligned}$$

Identity $1 \cdot [\gamma] = \gamma$

$$\begin{aligned}
&1 \cdot [\gamma]\\
&= [\phi^{-1}(1 \cdot \phi(\gamma) - (1-1)\phi(p))]\\
&= [\phi^{-1}(\phi(\gamma) - 0 \cdot \phi(p))]\\
&= [\phi^{-1}(\phi(\gamma))]\\
&= [\gamma]
\end{aligned}$$

Distributivity of scalar multiplication with respect to vector addition $a([\gamma]+[\eta]) = a[\gamma]+a[\eta]$

$$\begin{aligned}
&a([\gamma] + [\eta])\\
&= a[\phi^{-1}(\phi(\gamma) + \phi(\eta) - \phi(p))]\\
&= [\phi^{-1}(a\phi(\phi^{-1}(\phi(\gamma) + \phi(\eta) - \phi(p))) - (a-1)\phi(p))]
\end{aligned}$$

$$= [\phi^{-1}(a(\phi(\gamma) + \phi(\eta) - \phi(p)) - (a-1)\phi(p))]$$
$$= [\phi^{-1}(a\phi(\gamma) + a\phi(\eta) - a\phi(p) - (a-1)\phi(p))]$$

Distributivity of scalar multiplication with respect to field addition $(a+b)[\gamma] = a[\gamma] + b[\gamma]$

$$\begin{aligned}
(a+b)[\gamma] &= [\phi^{-1}((a+b)\phi(\gamma) - ((a+b)-1)\phi(p))] \\
&= [\phi^{-1}(a\phi(\gamma) + b\phi(\gamma) - a\phi(p) - b\phi(p) - \phi(p) + \phi(p) - \phi(p))] \\
&= [\phi^{-1}(a\phi(\gamma) - (a-1)\phi(p)) + (b\phi(\gamma) - (b-1)\phi(p)) - \phi(p))] \\
&= [\phi^{-1}(\phi(\phi^{-1}(a\phi(\gamma) - (a-1)\phi(p))) + \phi(\phi^{-1}(b\phi(\gamma) - (b-1)\phi(p))) - \phi(p))] \\
&= [\phi^{-1}(a\phi(\gamma) - (a-1)\phi(p))] + [\phi^{-1}(b\phi(\gamma) - (b-1)\phi(p))] \\
&= a[\gamma] + b[\gamma]
\end{aligned}$$

□

This kind of proof is one of the reason we want chart independent definitions like the one for T_pM. The reasons for not wanting this kind of proof is first of all that it is essentially symbol-soup, and secondly that it does not give any insight about the tangent space. However, this definition is isomorphic to the derivation definition, which is quite insightful and will be proven in the next bit.

2.4 Equivelance of definitions of between $\widetilde{T}_pM$ and T_pM

Now for the final section of the thesis it will be shown that equivelance class smooth curves is the same tangent space as derivations tangent space. For two mathematical definitions to be equivalent depends on the context of the objects being studied. For example for two topological spaces, in the context of topology, to be equivalent, it is only demanded that there exists an homeomorphism between them. When they are "homeomorphism equivalent" it is said that they are equal up to homeomorphism, implying a sort of hierarchy of equivalences. On the top of this hierarchy there is literal equivalence where statements such as $1 = 1$ belongs, and surprisingly also theorem 2.2.5. Another equivalence in the rhetorical hierarchy of equivalences is isomorphism which are bijective structure preserving map. It will be shown that T_pM and $\tilde{T}_pM$ is equal up to isomorphism. What is meant by structure preserving between vector spaces is that the bijective map is also linear.

Definition 2.4.1. Let $F : \widetilde{T}_pM \to T_pM$ defined by $F([\gamma]) := D_{[\gamma]}f = \frac{\partial}{\partial t}(f \circ \gamma)|_0$

Since the composition $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is a function function from real to real, how is it corresponding to derivations? Well the intuition is that since we take the composite with a curve we get the direction derivative in that direction.
Need to check that it does not depend on the representative of the class.

Lemma 2.4.1. *$\gamma, \eta \in [\gamma] \implies F([\gamma]) = F([\eta])$, does not depend on representative of class*

Proof.

$$F([\gamma]) = F([\eta]) \overset{\text{definition}}{\Longleftrightarrow}$$
$$\frac{\partial}{\partial t}(f \circ \gamma)|_0 = \frac{\partial}{\partial t}(f \circ \eta)|_0 \iff$$
$$\frac{\partial}{\partial t}(f \circ \phi^{-1} \circ \phi \circ \gamma)|_0 = \frac{\partial}{\partial t}(f \circ \phi^{-1} \circ \phi \circ \eta)|_0 \overset{\text{Multivariable chain rule}}{\Longleftrightarrow}$$
$$\sum_{i=1}^{n} \frac{\partial f \circ \phi^{-1}}{\partial x^i} \frac{\partial(\phi \circ \gamma)^i}{\partial t}\Big|_0 = \sum_{i=1}^{n} \frac{\partial f \circ \phi^{-1}}{\partial x^i} \frac{\partial(\phi \circ \eta)^i}{\partial t}\Big|_0 \Longleftarrow$$
$$\frac{\partial(\phi \circ \gamma)^i}{\partial t}\Big|_0 = \frac{\partial(\phi \circ \eta)^i}{\partial t}\Big|_0, \forall i \leq n \iff \gamma, \eta \in [\gamma]$$

□

The following lemma is essential in the idea for this isomorphism. Which is that we can rewrite any directional derivative into a derivative composite with a curve going in the direction of a given vector at the point $\hat{p}$.

Lemma 2.4.2.

$$\vec{v} \bullet \nabla \hat{f}|_{\hat{p}} = \frac{\partial}{\partial t} f \circ \gamma_{\vec{v}}(t)\Big|_{t=0}$$

where $\gamma_{\vec{v}}(t) = \phi^{-1}(\phi(p) + t\vec{v})$

Proof. Recalling that $\vec{v} \bullet \nabla \hat{f}|_{\hat{p}} = \nabla_{\vec{v}} \hat{f}|_{\hat{p}}$
Writing out the definitions

$$\nabla_{\vec{v}} \hat{f}|_{\hat{p}} = \lim_{t \to 0} \frac{\hat{f}(\vec{x} + t \cdot \vec{v}) - \hat{f}(\vec{x})}{h}|_{\hat{p}} = \lim_{t \to 0} \frac{f(\phi^{-1}(\phi(p) + t\vec{v})) - f(\phi^{-1}(\phi(p))}{t}$$

Now we see that $\gamma_{\vec{v}}(0) = p$

$$\lim_{t \to 0} \frac{f(\phi^{-1}(\phi(p) + t\vec{v})) - f(\phi^{-1}(\phi(p)))}{t} = \lim_{t \to 0} \frac{f(\gamma_{\vec{v}}(t)) - f(\gamma_{\vec{v}}(0))}{t} = \frac{\partial}{\partial t} f \circ \gamma_{\vec{v}}(t)\Big|_{t=0}$$

□

Lemma 2.4.3. *F is invertible and the inverse is $F^{-1}(\vec{v} \bullet \nabla \hat{f}|_{\hat{p}}) = [\gamma_{\vec{v}}]$*

Proof. $F(F^{-1}(D_{\vec{v}})) = F([\gamma_{\vec{v}}]) = \frac{\partial}{\partial t}(f \circ \gamma_{\vec{v}})|_0 \overset{\text{Lemma 2.4.2}}{=} D_{\vec{v}}$
$F^{-1}(F([\gamma])) = F^{-1}(\frac{\partial}{\partial t}(f \circ \gamma_{\vec{v}})|_0)$, because for all equivelance classes we have that $\frac{\partial}{\partial t}(\phi \circ \gamma)|_0 = \vec{v} = \frac{\partial}{\partial t}(\phi \circ \gamma_{\vec{v}})|_0$. $= F^{-1}(\frac{\partial}{\partial t}(f \circ \gamma_{\vec{v}})|_0) \overset{\text{Lemma 2.4.2}}{=} F^{-1}(\vec{v} \bullet \nabla \hat{f}|_{\hat{p}}) = [\gamma_{\vec{v}}]$ □

We are now ready to prove the main result.

Theorem 2.4.4. *F is an isomorphism*

Proof. It follows from that F is invertible (lemma 2.4.3) that it is bijective.
We will show in two steps that $F(a[\gamma] + b[\eta]) = aF([\gamma]) + bF([\eta])$. First $F(a[\gamma]) = aF([\gamma])$

and then $F([\gamma]+[\eta]) = F([\gamma]) + F([\eta])$

$$\begin{aligned}
F([\gamma]+[\eta]) &= F([\gamma]) + F([\eta]) \iff \\
F^{-1}(F([\gamma]+[\eta])) &= F^{-1}(F([\gamma]) + F([\eta])) \iff \\
[\gamma]+[\eta] &= F^{-1}(F([\gamma]) + F([\eta])) \\
&= F^{-1}(\frac{\partial}{\partial t}(f\circ\gamma_{\vec{v}})|_0 + \frac{\partial}{\partial t}(f\circ\eta_{\vec{w}}|_0)) \\
&= F^{-1}(\vec{v}\bullet\nabla\hat{f}|_{\hat{p}} + \vec{w}\bullet\nabla\hat{f}|_{\hat{p}}) \\
&= F^{-1}((\vec{v}+\vec{w})\bullet\nabla\hat{f}|_{\hat{p}}) \\
&= [\gamma_{\vec{v}+\vec{w}}] = [\phi^{-1}(\phi(p)+t(\vec{v}+\vec{w}))]
\end{aligned}$$

$\phi^{-1}(\phi(p)+t(\vec{v}+\vec{w})) \sim \phi^{-1}(\phi(\gamma)+\phi(\eta)-\phi(p)) \in [\gamma]+[\eta]$, because the following computation with the equivalence relation that is given.

$$\begin{aligned}
\frac{\partial}{\partial t}(\phi\circ\phi^{-1}(\phi(p)+t(\vec{v}+\vec{w})))|_0 &= \frac{\partial}{\partial t}(\phi\circ\phi^{-1}(\phi(\gamma)+\phi(\eta)-\phi(p)))|_0 \impliedby \\
\frac{\partial}{\partial t}(\phi(p)+t(\vec{v}+\vec{w}))|_0 &= \frac{\partial}{\partial t}\phi(\gamma)+\phi(\eta)-\phi(p))|_0 \impliedby \\
\frac{\partial}{\partial t}(t(\vec{v}+\vec{w}))|_0 &= \frac{\partial}{\partial t}\phi(\gamma)+\phi(\eta))|_0 \impliedby \\
\vec{v}+\vec{w} &= \frac{\partial}{\partial t}\phi(\gamma)+\phi(\eta))|_0 \impliedby \\
[\gamma_{\vec{v}}] \text{ such that } \vec{v} &= \frac{\partial}{\partial t}\phi\circ\gamma|_0 \text{ and } [\eta_{\vec{w}}] \text{ such that } \vec{w} = \frac{\partial}{\partial t}\phi\circ\eta|_0
\end{aligned}$$

So we have that $[\gamma]+[\eta] = [\phi^{-1}(\phi(\gamma)+\phi(\eta)-\phi(p))] = [\phi^{-1}(\phi(p)+t(\vec{v}+\vec{w}))] = [\gamma_{\vec{v}+\vec{w}}]$ and therefore $F([\gamma]+[\eta]) = F([\gamma]) + F([\eta])$

$$\begin{aligned}
F(a[\gamma]) &= aF([\gamma]) \impliedby \\
[\phi^{-1}(a\phi(\gamma)-(a-1)\phi(p))] &= F^{-1}(aF([\gamma])) \\
&= F^{-1}(a\frac{\partial}{\partial t}(f\circ\gamma)|_0) \\
&= F^{-1}((a\cdot\vec{v})\bullet\nabla\hat{f}|_{\hat{p}}) \\
&= [\gamma_{a\vec{v}}] \\
&= [\phi^{-1}(\phi(p)+t(a\cdot\vec{v}))]
\end{aligned}$$

The following computation shows that $\phi^{-1}(\phi(\gamma)+\phi(\eta)-\phi(p)) \sim \phi^{-1}(\phi(p)+t(\vec{v}+\vec{w}))$

$$\frac{\partial}{\partial t}(\phi\circ\phi^{-1}(a\phi(\gamma)-(a-1)\phi(p)))|_0 = \frac{\partial}{\partial t}(\phi\circ\phi^{-1}(\phi(p)+t(a\cdot\vec{v})))|_0 \impliedby$$

$$\frac{\partial}{\partial t}(a\phi(\gamma) - (a-1)\phi(p))|_0 = \frac{\partial}{\partial t}(\phi(p) + t(a \cdot \vec{v}))|_0 \impliedby$$
$$a\frac{\partial}{\partial t}(\phi(\gamma))|_0 = a \cdot \vec{v} \impliedby$$
$$\frac{\partial}{\partial t}(\phi(\gamma))|_0 = \vec{v}$$

From this it follows that $F(a[\gamma]) = aF([\gamma])$
Which proves the main result. $\square$

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